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LETTER TO THE EDITOR

On the renormalisation of the three-dimensional O(N) σ model

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Abstract. The renormalisation constants for the O(N) σ model in three dimensions are correctly computed within the large N expansion using dimensional regularisation.

Three-dimensional σ models and various four-fermi models (Rosenstein *et al* 1989a, b) provide an interesting class of field theories which are perturbatively non-renormalisable but which are renormalisable within the non-perturbative large-N expansion. Indeed the σ models on O(N) (Aref'eva 1977, 1979, Rosenstein *et al* 1989) and $\mathcal{CP}(N)$ (Aref'eva and Azakov 1980, Cant and Davis 1980), and its supersymmetric version (Aref'eva and Azakov 1980) have all been examined in great detail. Unlike their two-dimensional counterparts they possess a two-phase structure, where the critical coupling appears as a non-trivial zero of the β-function. In the lower phase the particles are massless and, in the case of O(N), the O(N) symmetry is broken, whilst in the upper phase mass is gained through dynamical symmetry breaking and the symmetry is restored. Indeed in this upper phase, the three-dimensional models share essentially all the properties of two dimensions, except asymptotic freedom. As it is well known that Einstein gravity possesses non-renormalisable interactions, one may hope it could be rendered finite through some non-perturbative approach analogous to the large-N expansion. Current interest in three-dimensional nonlinear σ models, though, is due to their relation to various statistical systems, such as models which describe ferromagnets, superconductors (Stanley 1971) or superfluids (Aref'eva and Azakov 1980). It is known that the O(N) model is related to the Heisenberg model, and consequently scattering data has been used recently to fit to various three-dimensional σ model parameters (Chakravarty *et al* 1988, 1989). This has provided an interesting connection between theoretical models and experimental results.

The initial work in understanding the three-dimensional σ models from a field theory point of view, however, was carried out over a decade ago (Aref'eva 1977, 1979). The large-N perturbation theory for the O(N) model was first constructed by Aref'eva (1977). It turns out, however, that there was an error in the explicit renormalisation performed there. In particular, one graph was omitted from the vertex renormalisation, and its absence would lead to a violation of the constraint, which ensures the

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bosons lie on S^N, at the quantum level. So it is the aim of this letter to (correctly) compute the next-to-leading-order renormalisation constants within the large-N expansion, using dimensional regularisation. This will give an explicit illustration of the general structure of the model which we will also relate to other models.

First we recall the important features of the O(N) model which are relevant to our computation. The (bare) Lagrangian for the model is

$$L_0 = \tfrac{1}{2}(\partial n_0)^2 - \tfrac{1}{2}\lambda_0(n_0^2 - 1/g_0) \tag{1}$$

where the Lagrange multiplier field, λ_0, ensures the fields lie on the N-sphere, and in three dimensions the bare coupling, g_0, has dimensions of inverse mass. If one calculates at leading order in cut-off regularisation, and examines the expectation value of λ, the β-function is deduced as (Aref'eva 1977),

$$\beta(\tilde{g}) = \tilde{g}\left(1 - \frac{N\tilde{g}}{4\pi}\right) \tag{2}$$

where $\tilde{g}(\mu)$ is the dimensionless renormalised coupling constant, and μ is the renormalisation scale. A phase transition occurs at $\tilde{g}_c = 4\pi/N$. The lower phase consists of massless bosons, whilst in the upper phase, the bosons gain a mass $\mu(1 - \tilde{g}_c/\tilde{g})$. We will consider only the massive phase in the following. It possesses several features in common with the two-dimensional model, such as the development of a non-trivial propagator for the λ field, which is deduced by inverting the λ two-point function at leading order (Aref'eva 1977), i.e.

$$-\frac{2\mathrm{i}}{NJ(k^2)} \tag{3}$$

where

$$J(k^2) = -\frac{1}{4\pi}\left(\frac{4m^2}{-k^2}\right)^{1/2} \tan^{-1}\left(\frac{-k^2}{4m^2}\right)^{1/2} \tag{4}$$

in three dimensions, and we work in Minkowski space throughout. In order to examine the renormalisation of (1), we introduce renormalised quantities and renormalisation constants for the upper phase via

$$n_0 = nZ_n \qquad g_0 = gZ_g \qquad \lambda_0 = m^2 Z_m + \lambda Z_\lambda \tag{5}$$

where the latter expression introduces the boson mass. As graphs with four and six boson legs are also superficially divergent, we ought to include the operators $(n^2)^2$ and $(n^2)^3$ in the renormalised Lagrangian as counterterms. However, the presence of such terms would be incompatible with the quantised version of the constraint $n^2 = 1/g$ (Aref'eva 1979). This is preserved after quantisation because the Green functions involving only λ fields as external legs are superficially divergent only when there is one λ field. Thus λ retains its role as a Lagrange multiplier after quantisation. This property was first observed by Aref'eva (1979). So $(n^2)^2$ and $(n^2)^3$ counterterms are excluded and we comment further on this later. For next-to-leading-order calculations, we use the dimensional regularisation procedure of Aref'eva (1977). This entails computing d-dimensional integrals, but using (3), which can be expanded in powers of $(m^2/k^2)^{1/2}$, to enable the ultraviolet structure of the various integrals to be computed. The renormalisation constants will subsequently involve simple poles in ε, where $d = 3 - 2\varepsilon$.

The graphs which we consider at next-to-leading order are illustrated in figures 1-3, where the square blobs denote counterterms. The corrections to the boson propagator have been correctly computed by Aref'eva (1977), and we note

$$Z_n = 1 - \frac{4}{3N\pi^2\varepsilon} \qquad Z_n Z_m = 1 + \frac{4}{N\pi^2\varepsilon}\left(1 - \frac{16}{\pi^2}\right) \tag{6}$$

where the latter was not computed explicitly. For the vertex renormalisation of figure 2, we note that the second graph was omitted from the original analysis, and computing it, we find it is simply

$$-\frac{4\mathrm{i}}{N}\int_k\int_l \{J(k^2)(l^2-m^2)^2[(l-k)^2-m^2](k^2-m^2)\}^{-1}. \tag{7}$$

The l-integration is simplified by using the cutting rule of Rim and Weisberger (1984), which is in effect a Gauss recursion relation for the hypergeometric function, but in three dimensions:

$$\mathrm{i}(k^2-4m^2)\int_l \{(l^2-m^2)^2[(l-k)^2-m^2]\}^{-1} = J(0)-(d-3)J(k^2) \tag{8}$$

where the second term gives a finite contribution on completing the k-integration of (7). Thus with the additional divergent piece from the first graph of figure 2, we have

$$Z_n Z_\lambda = 1 + \frac{4}{N\pi^2\varepsilon}\left(1 - \frac{16}{\pi^2}\right) \tag{9}$$

Figure 1. Boson propagator corrections.

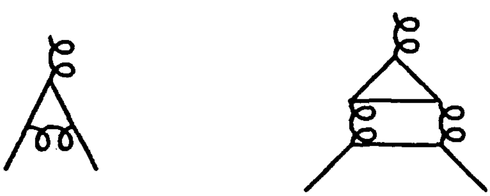

Figure 2. Vertex corrections.

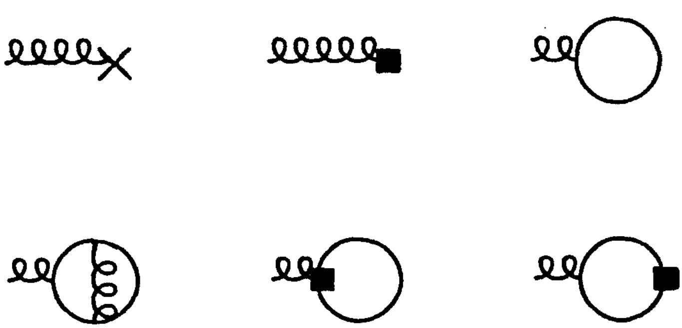

Figure 3. Vacuum expectation value of λ.

from which it emerges that

$$Z_\lambda = Z_m \tag{10}$$

at this order. Of course, if the second graph of figure 2 is omitted, then (10) does not follow. However, (10) must occur because an n^2 counterterm, in addition to that which occurs due to the development of a non-zero vacuum expectation value for the λ field, cannot be present for the same reason that $(n^2)^2$ and $(n^2)^3$ counterterms were excluded (Aref'eva 1979, Rosenstein *et al* 1989a) i.e. incompatibility with the constraint at the quantum level. Thus the model has only two independent renormalisation constants, and we note the two-dimensional model has a similar structure (Rim and Weisberger 1984, Davis *et al* 1989).

Finally, as the renormalisation constant (9) differs from that of Aref'eva (1977), we must reconsider the corrections to the vacuum expectation value of λ at next-to-leading order, since it occurs as a counterterm in the fifth graph of figure 3. Again the cutting rule (8) is used to simplify the fourth graph of figure 3. Thus we find,

$$Z_g^{-1} = 1 + \frac{4}{3\pi^3\varepsilon} \tag{11}$$

which differs from that obtained by Aref'eva (1977).

We close with several comments. Firstly, we have correctly computed the renormalisation constants of the model at next-to-leading order. We found only two independent renormalisation constants are required, consistent with Aref'eva (1979), Rosenstein *et al* (1989a). Secondly, we note the relation of our results with other three-dimensional models. In a similar calculation in the Gross–Neveu model, for instance, one finds the coupling constant renormalisation at next to leading order is $Z_g^{-1} = 1 - 4/(3\pi^3\varepsilon)$ (Gracey 1990). A similar feature occurs in the two-dimensional models (Rim and Weisberger 1984, Davis *et al* 1989), where, in dimensional regularisation, one finds $\ln\varepsilon$ type divergences in the coupling constant renormalisation, but they arise with differing signs in the two models. Accordingly such divergences are absent in the supersymmetric version (Davis *et al* 1989), since the $O(N)$ σ and Gross–Neveu models are the boson and fermion sectors respectively. In three dimensions a similar feature occurs in the upper massive phase. Thus in dimensional regularisation, we have $Z_g = 1$. An alternative way to view this is that in a cut-off regularisation of the supersymmetric model, the next-to-leading-order corrections to Z_g would be linear in the cut-off, Λ only, and the $\ln(\Lambda^2/m^2)$ divergences, which correspond to simple poles in ε, actually cancel. Finally, we note the next-to-leading-order corrections for (1) do not affect the general properties of the β-function (2). It still retains a non-zero fixed point, which using cut-off regularisation, and a particular choice of renormalisation condition (Cant and Davis 1980) remains at $g_c = 4\pi/N$.

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